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Reducible Veronese surfaces

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Abstract. Here we describe all degree $n + 3$ non-degenerate surfaces in $\mathbb{P}^{n+4}$, $n \geq 1$, connected in codimension 1, which may be isomorphically projected into $\mathbb{P}^4$. There are three of them. One is a suitable union of $n + 3$ planes (for all $n \geq 1$); it was discovered by Floystad. The other two are unions of a smooth quadric and two planes (only for $n = 1$).

Key words. Reducible surfaces, projectability.

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1 Introduction

Let $\mathbb{P}^N$ be the N-dimensional projective space on $\mathbb{C}$. For any integer $k \geq 0$, a reduced subvariety $V \subset \mathbb{P}^N$ of pure dimension is said to be connected in codimension k if for any closed subvariety $W \subset V$, such that $\mathrm{cod}_V(W) > k$, we have that $V \backslash W$ is connected. For any subvariety $V \subset \mathbb{P}^N$ and for any λ-dimensional linear subspace $\Lambda \subset \mathbb{P}^N$ we say that V projects isomorphically to Λ if there exists a linear projection $\pi_{\mathcal{L}} : \mathbb{P}^N \dashrightarrow \Lambda$, from a suitable $(N - \lambda - 1)$-dimensional linear space $\mathcal{L}$, disjoint from V, such that $\pi_{\mathcal{L}}(V)$ is isomorphic to V.

In this note we consider the following type of surface arising from the example described in Section 2.

Definition 1. For any positive integer $n \geq 1$, we will call *reducible Veronese surface* any algebraic surface $X \subset \mathbb{P}^{n+4}$ such that:

i) X is a non-degenerate, reduced, reducible surface of pure dimension 2;
ii) $\deg(X) = n + 3$, $\mathrm{cod}(X) = n + 2$, so that X is a minimal degree surface;
iii) $\dim[\mathrm{Sec}(X)] \leq 4$, so that it is possible to choose a generic linear space $\mathcal{L}$ of dimension $n - 1$ in $\mathbb{P}^{n+4}$ such $\pi_{\mathcal{L}}(X)$ is isomorphic to X, where $\pi_{\mathcal{L}}$ is the the rational projection $\pi_{\mathcal{L}} : \mathbb{P}^{n+4} \dashrightarrow \Lambda$ from $\mathcal{L}$ to a generic target $\Lambda \simeq \mathbb{P}^4$;

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iv) X is connected in codimension 1, i.e. if we drop any finite number (possibly 0) of points $Q_1, \dots, Q_r$ from X we have that $X \setminus \{Q_1, \dots, Q_r\}$ is connected;
v) X is a locally Cohen–Macaulay surface.

Remark 1. Actually v) implies iv) by Corollary 2.4 of [6]; however we think that it is more useful to give the above Definition 1 because condition iv) is crucial to get the classification.

In summary: we prove that there are exactly 3 types of reducible Veronese surfaces (see Proposition 2 and Theorems 2, 3, 4):

i) a suitable union of $n+3$ planes (for any integer $n \geq 1$) which sits as a linearly normal scheme in $\mathbb{P}^{n+4}$ (see Theorem 2 and Definition 2 for a precise description); these are the examples whose existence is proved in [5];
ii) two surfaces which are the union of a smooth quadric surface and two planes; each of these two examples sits as a linearly normal scheme in $\mathbb{P}^5$ (see Theorems 3 for their description).

We will use the following definitions:

$\langle V_1 \cup \dots \cup V_r \rangle$:	linear span in $\mathbb{P}^N$ of the subvarieties $V_i \subset \mathbb{P}^N$, $i = 1, \dots, r$;
$\mathrm{Supp}(V)$:	support of the subscheme $V \subset \mathbb{P}^N$;
$\mathrm{Sing}(V)$:	singular locus of the subscheme $V \subset \mathbb{P}^N$;
$\mathrm{Sec}(V)$:	$\overline{\left\{\bigcup_{v_1 \neq v_2 \in V} \langle v_1 \cup v_2 \rangle\right\}} \subset \mathbb{P}^N$ for any subvariety $V \subset \mathbb{P}^N$.

For any positive integer $d \geq 2$ a rational comb of degree d in $\mathbb{P}^N$ is the union of d lines $L_1, L_2, \dots, L_d \subset \mathbb{P}^N$ such that, for any $i \geq 2$, $L_i \cap L_1$ is a point, these $d-1$ points are distinct and, for any $j > i \geq 2$, $L_i \cap L_j = \emptyset$.

2 Floystad's example

In [5, Corollary 3], the author proves that, for any integer $n \geq 1$, there exists in $\mathbb{P}^4$ a monad of the following form:

$$\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus n+2} \longrightarrow \mathcal{O}_{\mathbb{P}^4}{}^{\oplus 2n+3} \longrightarrow \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus n}$$

whose homology is $\mathcal{I}_{S_n}(2)$ where S_n is a locally Cohen–Macaulay surface in $\mathbb{P}^4$. Moreover S_n is embedded in $\mathbb{P}^{n+4}$ as a linearly normal surface and S_n projects isomorphically to some suitable $\Lambda \subset \mathbb{P}^{n+4}$, $\Lambda \simeq \mathbb{P}^4$. For $n = 1$, S_1 is the usual (smooth) Veronese surface in $\mathbb{P}^5$; in contrast, S_n must be singular for $n \geq 2$.

If we call $\varphi_n : \mathcal{O}_{\mathbb{P}^4}{}^{\oplus 2n+3} \longrightarrow \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus n}$ we get the following exact sequences of sheaves and vector bundles over $\mathbb{P}^4$:

$$0 \longrightarrow \ker(\varphi_n) \longrightarrow \mathcal{O}_{\mathbb{P}^4}{}^{\oplus 2n+3} \longrightarrow \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus n} \longrightarrow 0$$
$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus n+2} \longrightarrow \ker(\varphi_n) \longrightarrow \mathcal{I}_{S_n}(2) \longrightarrow 0.$$

Now it is easy to calculate $\chi[\mathcal{O}_{S_n}(t)] = \binom{t+4}{4} - \chi[\mathcal{I}_{S_n}(t)] = (\frac{n+3}{2})t^2 + (\frac{n+5}{2})t + 1$, so that $\deg(S_n) = n+3$ and S_n is a minimal degree surface in $\mathbb{P}^{n+4}$ for any $n \geq 1$.

When $n = 2$, by a computer algebra system as Macaulay, it is easy to get a set of generators for the ideal of a generic S_2 in $\mathbb{P}^6$. In fact, by choosing a random $(2, 7)$ matrix M of linear forms we have a map as φ_n and, by calculating the higher syzygies of M, we get a free resolution for $\ker(\varphi_n)$ and a commutative diagram as follows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4} & \longrightarrow & \ker(\varphi_2) & \longrightarrow & \mathcal{I}_{S_2}(2) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4} & \longrightarrow & \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 5} \oplus \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 10} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 20} & & & & \\
 & & & & \uparrow & & & & \\
 & & & & \vdots & & & &
\end{array}$$

By choosing another random $(5, 4)$ matrix N of constants, in order to get a map $\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4} \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 5}$, ($\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4} \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 10}$ is the zero map) and by using the mapping cone technique, we have that the ideal I_{S_2} in $\mathbb{P}^6$ of a generic surface S_2 is generated by one cubic and ten quartics. S_2 has codimension 4, degree 5 and (arithmetic) sectional genus 0. Alternatively, one can also choose 4 generic sections of the rank 5 vector bundle $\ker(\varphi_2) \otimes \mathcal{O}_{\mathbb{P}^4}(1)$ by giving a random $(5, 4)$ matrix of constants N': in this case S_2 is the degeneracy locus in $\mathbb{P}^6$ of these sections; if $N' = N$ we get exactly the same set of generators for I_{S_2}.

By knowing a set of generators for I_{S_2} it is, more or less, easy to see that the generic S_2 is given by 5 planes $\Pi_0, \Pi_1, \ldots, \Pi_4$ such that: $\Pi_0 \cap \Pi_i := L_i$ is a line for $i = 1, \ldots, 4$; $\Pi_i \cap \Pi_j := Q_{ij}$ is a point of Π_0 for $i, j = 1, \ldots, 4$, $i \neq j$, and the lines L_i are in general position on Π_0. The generic hyperplane section of S_2 is a rational comb of degree 5 given by a line l_0 on Π_0 and four other lines $l_i, i = 1, \ldots, 4$, $l_i \in \Pi_i$, $l_i \cap l_j = \emptyset$ for $i \neq j$, intersecting l_0 at one point. $\mathrm{Sec}(S_2)$ is the union of a finite number of linear spaces of dimension 2 (Π_i, $i = 0, \ldots, 4$), 3 ($\langle \Pi_0 \cup \Pi_i \rangle$, $i = 1, \ldots, 4$) or 4 ($\langle \Pi_i \cup \Pi_j \rangle$, $i, j = 1, \ldots, 4, i \neq j$) so that it is possible to choose a generic line $\mathcal{L}$ in $\mathbb{P}^6$, $\mathcal{L} \cap \mathrm{Sec}(S_2) = \emptyset$, and to project S_2, from $\mathcal{L}$ to a generic $\Lambda \simeq \mathbb{P}^4$, in such a way that the projection of S_2 is isomorphic to S_2.

The above concrete construction of S_2 suggests to define a family of completely reducible surfaces having the same properties.

Definition 2. For any positive integer $n \geq 1$, let us choose a plane Π_0 and $n + 2$ distinct points $P_1, \ldots, P_{n+2}$ in general position in $\mathbb{P}^{n+4}$, so that $\langle \Pi_0 \cup P_1 \cup \cdots \cup P_{n+2} \rangle = \mathbb{P}^{n+4}$. Let us choose $n + 2$ planes Π_i, $i = 1, \ldots, n + 2$, $P_i \in \Pi_i$, such that $\Pi_i \cap \Pi_0$ is a line L_i and the $n + 2$ lines L_i are in general position on Π_0 (i.e. that the curve given by their union has no triple points). Let us call Σ_n any surface in $\mathbb{P}^{n+4}$ which is the union $\Pi_0 \cup \Pi_1 \cdots \cup \Pi_{n+2}$.

Proposition 1. *The previously defined surfaces Σ_n, $n \geq 1$, are reducible Veronese surfaces according to Definition* 1.

Proof. i), ii), iii), iv) follow directly from the definition; note that $\mathrm{Sec}(\Sigma_n)$ is the union of a finite number of linear spaces of dimension $2, 3, 4$.

Concerning v), let us remark that for any singular point $P \in \Sigma_n$ its local ring is isomorphic either to the local ring at $(0,0,0)$ of the affine variety $\{xy = 0\}$ in $\mathbb{A}^3(\mathbb{C})$, or to the local ring at $(0,0,0,0)$ of the affine variety $\{x = y = 0\} \cup \{z = w = 0\} \cup \{x = z = 0\} = \{x^2z = xz^2 = x^2w = xzw = xyz = yz^2 = xyw = yzw = 0\}$ in $\mathbb{A}^4(\mathbb{C})$. They are, up to isomorphisms, the same local rings of the singular points of S_2 and we know that S_2 is a locally Cohen–Macaulay surface by Corollary 3 of [5]. □

To prove that Σ_n are locally Cohen–Macaulay we could also use a slightly different version of the following lemma which will be useful at the end of the paper.

Lemma 1. *Let $X \subset \mathbb{P}^5$ be a non-degenerate surface such that $X = Q \cup X_1 \cup X_2$, where Q is a smooth quadric, X_1 and X_2 are planes, and either X_1 and X_2 cut Q along two lines intersecting at a point $P = X_1 \cap X_2$ or Q, X_1, X_2 intersect transversally along a unique line $L = Q \cap X_1 \cap X_2$. Then X is a locally Cohen–Macaulay surface.*

Proof. Let us consider the first case. Obviously we have to check the property only at P. Let R be the local ring of X at P and let m be its maximal ideal. We have $\mathrm{height}(m) = 2$, so that we have to prove that $\mathrm{depth}(m) = 2$. As X is reduced and $\dim(X) \geq 1$ we know that $\mathrm{depth}(m) \geq 1$. A generic hyperplane section of X not passing through P cuts X along a reducible curve $Y = C \cup L_1 \cup L_2$, where C is a smooth conic and L_1, L_2 are two disjoint lines intersecting C transversally at two different points. Y is reduced, connected and its arithmetic genus $p_a(Y)$ is 0. Let H be a generic hyperplane section of X passing through P; now $H \cap X := Y_P$ is reducible as the union of a smooth conic C_P and two distinct lines intersecting C_P transversally at P. H gives rise to a non-zero divisor element $\alpha \in m$ because X has pure dimension 2. Now let us remark that $p_a(Y_P) = 0$, so that Y_P has no embedded components at $P = \mathrm{Sing}(Y_P)$, otherwise $p_a(Y_P) < p_a(Y)$. Hence there is at least a non-zero divisor element $\beta \in m \setminus (\alpha)$ and (α, β) is a regular sequence for m, so that $\mathrm{depth}(m) \geq 2$. As $\mathrm{depth}(m) \leq \mathrm{height}(m) = 2$ we are done.

In the second case we can argue as in the previous one for all points $P \in L$. □

Remark 2. It is easy to see that the generic section of Σ_n is a rational comb, quite exactly as in the case of S_2 (which is in fact an example of Σ_2), so that $p_a(\Sigma_n) = 0$, but we will not consider this property in the sequel.

Now it is very natural to ask if the surfaces Σ_n are the only existing reducible Veronese surfaces in our sense. The answer to this question is the aim of the following sections. Moreover we will prove that any generic S_n is a surface Σ_n for $n \geq 2$, see Remark 3. To show that the matter is in fact very intricate, let us consider the following:

Example 1. Let $X = Q \cup \Pi_1 \cup \Pi_2 \cup \Pi_3 \subset \mathbb{P}^6$, where Q is a smooth quadric of $\mathbb{P}^3$ and any Π_i is a generic plane such that, if we call the three points $P_{ij} := \Pi_i \cap \Pi_j$, we have: $P_{ij} \notin \Pi_k$ for $k \neq i, j$, $P_{ij} \notin Q$, but $P_{ij} \in \langle Q \rangle$. Then X is non-degenerate, $\deg(X) = 5$, $\dim[\mathrm{Sec}(X)] \leq 4$, but X is not connected in codimension 1, for instance because $X \backslash \{P_{12} \cup P_{23} \cup P_{31}\}$ is not connected.

3 Xambò's result and applications

In [7] Xambò proves the following result:

Theorem 1. *Let $V = V_1 \cup \cdots \cup V_r \subset \mathbb{P}^N$ be a non-degenerate, reducible, reduced, surface of pure dimension 2, whose irreducible components are $V_1, \ldots, V_r$. Assume that V is connected in codimension 1 and that it has minimal degree. Then*

- *any irreducible component V_i of dimension 2 of V is a surface of minimal degree in its span $\langle V_i \rangle$;*
- *there is at least an ordering $V_1, V_2, \ldots, V_r$ such that, for any $j = 2, \ldots, r$,*

$$V_j \cap (V_1 \cup \cdots \cup V_{j-1}) = \langle V_j \rangle \cap \langle V_1 \cup \cdots \cup V_{j-1} \rangle$$

  and this intersection is always a line.

Proof. The theorem is a simple consequence of Theorem 1 of [7]. □

Corollary 1. *Let $\Pi_1, \Pi_2, \ldots, \Pi_r$ be a set of ordered planes in some $\mathbb{P}^N$ such that:*

i) $\langle \Pi_1 \cup \Pi_2 \cup \cdots \cup \Pi_r \rangle = \mathbb{P}^N$;
ii) *for any* $j \geq 2$, $\dim(\Pi_j \cap \langle \Pi_1 \cup \cdots \cup \Pi_{j-1} \rangle) = 1$.

Then $X := \Pi_1 \cup \Pi_2 \cup \cdots \cup \Pi_r$ is a non-degenerate surface in $\mathbb{P}^N$, of minimal degree, connected in codimension 1.

Proof. The corollary follows from the remark after Theorem 1 of [7, p. 151]. □

Corollary 2. *Let V be any surface as in Theorem 1. Then for any pair of irreducible components V_j, $V_k \subset V$ we have only three possibilities:*

- $V_j \cap V_k = \emptyset$
- $V_j \cap V_k$ *is a point*
- $V_j \cap V_k$ *is a line.*

Proof. Let us assume that $V_j \cap V_k \neq \emptyset$ and that $k > j$ in the existing ordering of the components of V considered by Theorem 1. Then $V_j \cap V_k \subseteq V_k \cap (V_1 \cup \ldots V_j \cup \cdots \cup V_{k-1})$ which is a line, as a scheme, because it is the intersection of two linear spaces in $\mathbb{P}^N$. By Theorem 0.4 of [4] V is small according to the definition of [4, p. 1364] hence $V_j \cap V_k = \langle V_i \rangle \cap \langle V_j \rangle$ is a linear space by Proposition 2.4 of [4]. As $V_j \cap V_k$ is contained in a line Corollary 2 follows. □

Lemma 2. *Let $X \subset \mathbb{P}^{n+4}$ be a reducible Veronese surface, according to Definition 1, for some $n \geq 1$. Then:*

i) *any connected surface $Y \subset X$ can be isomorphically projected in $\mathbb{P}^4$;*
ii) *for any pair of irreducible components X_j and X_k of X we have $X_j \cap X_k \neq \emptyset$.*

Proof. As X is a reducible Veronese surface there exists a projection $\pi_{\mathcal{L}} : \mathbb{P}^{n+4} \dashrightarrow \Lambda$, from a suitable linear space $\mathcal{L}$ to a suitable linear space $\Lambda \subset \mathbb{P}^{n+4}$, $\Lambda \simeq \mathbb{P}^4$, such that $\pi_{\mathcal{L}}(X) \simeq X$. This implies that, for any $i = 1, \dots, r$, $\pi_{\mathcal{L}}(X_i) \simeq X_i$, and, for any pair $X_j, X_k \subset X$, $\pi_{\mathcal{L}}(X_j) \cap \pi_{\mathcal{L}}(X_k) \simeq X_j \cap X_k$. Hence for any surface $Y \subset X$ we have $\pi_{\mathcal{L}}(Y) \simeq Y$ and $\pi_{\mathcal{L}}(X_j) \cap \pi_{\mathcal{L}}(X_k)$, being the intersection of two surfaces in $\mathbb{P}^4$, cannot be empty, so that $X_j \cap X_k$ cannot be empty too. □

Lemma 3. *Let $X \subset \mathbb{P}^{n+4}$ be a reducible Veronese surface, according to Definition 1, for some $n \geq 1$. Let P be a singular point of X and let $X_1^P, \dots, X_s^P$ be the irreducible components of X containing P with $s \geq 2$. For any $i = 1, \dots, s$ let T_i be the tangent space of X_i^P at P in $\langle X_i^P \rangle$ and let us assume that the natural ordering of $X_1^P, \dots, X_s^P$ is coherent with the ordering given by Theorem 1. Then, for any $j \geq 2$, $T_j \nsubseteq \langle T_1 \cup \cdots \cup T_{j-1} \rangle$ and $\dim[T_j \cap \langle T_1 \cup \cdots \cup T_{j-1} \rangle] \leq 1$.*

Proof. By contradiction, let us assume that $T_j \subseteq \langle T_1 \cup \cdots \cup T_{j-1} \rangle$, hence $T_j \subseteq T_j \cap \langle T_1 \cup \cdots \cup T_{j-1} \rangle \subseteq \langle X_j^P \rangle \cap \langle X_1^P \cup \cdots \cup X_{j-1}^P \rangle$. As we are assuming that the natural ordering of $X_1^P, \dots, X_s^P$ is coherent with the ordering given by Theorem 1, we have that $\dim[\langle X_j^P \rangle \cap \langle X_1^P \cup \cdots \cup X_{j-1}^P \rangle] \leq 1$. Moreover $\dim(T_j) = 2$ if P is a smooth point of X_j^P and $\dim(T_j) = 3$ if P is a singular point of X_j^P; in fact by Theorem 1 we know that every X_j is an irreducible, reduced, surface of minimal degree in its span and from the well known classification of these surfaces (see for instance Theorem 0.1 of [4]) we have that X_j is singular if and only if it is a rank 3 quadric. So that in any case we get a contradiction. By the way we have also proved that $\dim[T_j \cap \langle T_1 \cup \cdots \cup T_{j-1} \rangle] \leq 1$. □

Lemma 4. *Let $X \subset \mathbb{P}^{n+4}$ be a reducible Veronese surface, according to Definition 1, for some $n \geq 1$. Let P be any point of X and let $X_1^P, \dots, X_s^P$ be the irreducible components of X containing P, $s \geq 1$. For any $i = 1, \dots, s$ let T_i be the tangent space of X_i^P at P in $\langle X_i^P \rangle$ and let $\mathbb{T}_P := \bigcup_{i=1}^s T_i$. Then $\dim(\langle \mathbb{T}_P \rangle) \leq 4$.*

Proof. If $s = 1$ we have that $\langle \mathbb{T}_P \rangle = T_1$ and $\dim(T_1) \leq 3$ as in the proof of Lemma 3. If $s \geq 2$, $\mathbb{T}_P$ is the union of s linear spaces, of dimensions 2 or 3, passing through P according a certain configuration $\mathcal{C}_P \subset \mathbb{P}^{n+4}$. By contradiction, let us assume that $\dim(\langle \mathbb{T}_P \rangle) \geq 5$. Let $\pi_{\mathcal{L}} : \mathbb{P}^{n+4} \dashrightarrow \Lambda$ be any linear projection, from a suitable $(n-1)$-dimensional linear space $\mathcal{L}$ to a suitable $\Lambda \subset \mathbb{P}^{n+4}$, $\Lambda \simeq \mathbb{P}^4$, such that $\pi_{\mathcal{L}}(X)$ is isomorphic to X, hence $\pi_{\mathcal{L}}(\mathcal{C}_P)$ is isomorphic to $\mathcal{C}_P$. As $\dim(\langle \mathbb{T}_P \rangle) \geq 5$ there is a non-empty linear space $\mathcal{L}' := \mathcal{L} \cap \langle \mathbb{T}_P \rangle$ such that $\pi_{\mathcal{L}}(\mathcal{C}_P) = \pi_{\mathcal{L}'}(\mathcal{C}_P)$ where $\pi_{\mathcal{L}'} : \langle \mathbb{T}_P \rangle \dashrightarrow \Lambda$. But, as $\dim(\Lambda) < \dim(\langle \mathbb{T}_P \rangle)$, it is not possible that $\pi_{\mathcal{L}'}(\mathcal{C}_P) \simeq \mathcal{C}_P$, otherwise isomorphic configurations of linear spaces would have linear spans of different dimensions, so that we get a contradiction. □

Lemma 5. *Let V and W be two irreducible surfaces of $\mathbb{P}^N$ such that $V \cap W = \langle V\rangle \cap \langle W\rangle$ is a line L. Let us assume that each of V and W is a smooth rational scroll of degree 3 in $\mathbb{P}^4$, or a smooth quadric in $\mathbb{P}^3$, or a rank 3 quadric in $\mathbb{P}^3$. Then $\dim[\mathrm{Join}(V, W)] = 5$ unless V and W are both rank 3 quadrics, having the same vertex.*

Proof. Let us recall that $\mathrm{Join}(V, W) := \overline{\left\{\bigcup_{v\in V\setminus L, w\in W\setminus L} \langle v \cup w\rangle\right\}} \subset \mathbb{P}^N$. Let $\mathcal{U} \subset \mathrm{Join}(V, W)$ be the open set $\left\{\bigcup_{v\in V\setminus L, w\in W\setminus L} \langle v\cup w\rangle\right\}$. That suffices to show that $\dim(\mathcal{U}) = 5$.

Let p be a generic point of $\mathcal{U}$, hence $p \in \langle v\cup w\rangle$ for two generic points $v \in V\setminus L, w \in W\setminus L$ and we claim that, in our assumptions, $\langle v \cup w\rangle$ is the only line of $\mathcal{U}$ containing p. By contradiction, let us suppose that there exists another line $\langle v' \cup w'\rangle \neq \langle v \cup w\rangle$, with $v' \in V\setminus L, w' \in W\setminus L$, such that $p \in \langle v' \cup w'\rangle$. Then the two lines $\langle v \cup v'\rangle$ and $\langle w \cup w'\rangle$ intersect at a point $q \in L = \langle V\rangle \cap \langle W\rangle$. But our surfaces have no trisecant lines and, for generic points $v \in V\setminus L, w \in W\setminus L$, it is not possible that $\langle v\cup v'\rangle \cap \langle w\cup w'\rangle$ is a point of L, when $\langle v\cup v'\rangle \subset V$ and $\langle w\cup w'\rangle \subset W$, unless V and W are rank 3 quadrics of common vertex P. In this case there are infinitely many pairs of points $v' \in V\setminus L, w' \in W\setminus L$ such that $\langle v \cup v'\rangle \cap \langle w \cup w'\rangle = P$ (and $\dim[\mathrm{Join}(V, W)] = 4$). So that the claim is proved. Now we can define a rational map $s : \mathcal{U} \longrightarrow G(1, N)$, the Grassmannian of lines in $\mathbb{P}^N$, such that $s(p) = \langle v \cup w\rangle$. Of course the generic fibre of s has dimension 1 and $\dim(\mathrm{Im}(s)) = 4$, so that $\dim(\mathcal{U}) = 5$. □

From Theorem 1, and from the previous lemmas we get the following:

Proposition 2. *Every reducible Veronese surface $X \subset \mathbb{P}^{n+4}$, according to Definition 1, can be only the union $X = X_1 \cup\cdots\cup X_r$ of irreducible, reduced surfaces of the following types:*

- *planes*
- *smooth quadrics of $\mathbb{P}^3$*
- *quadrics of $\mathbb{P}^3$ having rank 3 (quadric cones for simplicity).*

Moreover only one irreducible surface of degree 2 can be contained in X.

Proof. From Theorem 1 we know that $X = X_1 \cup \cdots \cup X_r$ and that every X_j is an irreducible, reduced, surface of minimal degree in its span. From the well known classification of irreducible, reduced surfaces of minimal degree (see Theorem 0.1 of [4]), we have that every X_j is a surface as above or it is a smooth Veronese surface, a smooth rational scroll of degree 4 in $\mathbb{P}^5$, a smooth rational scroll of degree 3 in $\mathbb{P}^4$.

As any surface X_j contains a line by Theorem 1, none of them can be a smooth Veronese surface. The secant variety of a smooth rational scroll of degree 4 has dimension 5, so that X cannot contain such surfaces by condition iii) of Definition 1.

Let us consider a smooth rational scroll of degree 3 and let us assume, by contradiction, that it is a component of X, say X_j. Let X_k be any other component of X, different from X_j, and suppose that X_k is not a plane. As X is a reducible Veronese surface there exists a projection $\pi_{\mathcal{L}} : \mathbb{P}^{n+4} \dashrightarrow \Lambda$, from a suitable linear space $\mathcal{L}$ to a suitable $\Lambda \simeq \mathbb{P}^4$, such that $\pi_{\mathcal{L}}(X) \simeq X$. This implies that, for any $i = 1, \ldots, r$, $\pi_{\mathcal{L}}(X_i) \simeq X_i$, and $\pi_{\mathcal{L}}(X_j) \cap \pi_{\mathcal{L}}(X_k) \simeq X_j \cap X_k$. Recall that $\pi_{\mathcal{L}}(X_j) \cap \pi_{\mathcal{L}}(X_k)$ is the intersection of two surfaces in $\mathbb{P}^4$ and that, by assumption, $\pi_{\mathcal{L}}(X_j)$ is a smooth rational scroll of degree 3 and $\pi_{\mathcal{L}}(X_k)$ is another rational scroll of degree 3 or a quadric cone or a smooth quadric. Let us examine these possibilities.

If $\pi_{\mathcal{L}}(X_k)$ is another rational scroll of degree 3 then, by Lemma 2, $\pi_{\mathcal{L}}(X_j) \cap \pi_{\mathcal{L}}(X_k)$ cannot be empty, hence $\dim[\pi_{\mathcal{L}}(X_j) \cap \pi_{\mathcal{L}}(X_k)] \geq 0$. If $\dim[\pi_{\mathcal{L}}(X_j) \cap \pi_{\mathcal{L}}(X_k)] = \dim(X_j \cap X_k) = 0$, then $\deg[\pi_{\mathcal{L}}(X_j) \cap \pi_{\mathcal{L}}(X_k)] = 9$ and this is not possible by Corollary 2. Hence $\dim[\pi_{\mathcal{L}}(X_j) \cap \pi_{\mathcal{L}}(X_k)] = \dim(X_j \cap X_k) \geq 1$ and, by Corollary 2, $X_j \cap X_k = \langle X_j \rangle \cap \langle X_k \rangle$ is a line, so that $\dim[\mathrm{Join}(X_j, X_k)] = 5$ by Lemma 5, and $\dim[\mathrm{Sec}(X_j \cup X_k)] \geq 5$. This implies $\dim[\mathrm{Sec}(X)] \geq 5$, giving a contradiction with Definition 1 iii).

If $\pi_{\mathcal{L}}(X_k)$ is a quadric cone or a smooth quadric we can argue in the same way.

Now let us assume that $X_k \simeq \pi_{\mathcal{L}}(X_k)$ is a plane. By the above arguments, the only possibility is that the plane $\pi_{\mathcal{L}}(X_k)$ cuts $\pi_{\mathcal{L}}(X_j)$ along a line l, but also this case can be excluded, in fact we can consider a generic hyperplane H of Λ containing the plane $\pi_{\mathcal{L}}(X_k)$, the intersection $H \cap \pi_{\mathcal{L}}(X_j)$ is the union of l and of a smooth conic Γ. As Γ and $\pi_{\mathcal{L}}(X_k)$ are contained in $H \simeq \mathbb{P}^3$ their intersection cannot be empty, so that $\mathrm{Supp}[\pi_{\mathcal{L}}(X_j) \cap \pi_{\mathcal{L}}(X_k)]$ is not contained in a line and we have a contradiction with Corollary 2.

After proving that none of the irreducible components of X can be a rational scroll of degree 3, let us exclude that X has two (or more) components of degree 2, i.e. smooth quadrics or quadric cones. By contradiction, let us assume that X contains two irreducible components of degree 2, say X_j and X_k as before, and suppose that they are not both quadric cones with the same vertex. Then we can repeat the same argument, with the only difference that now $\langle X_j \rangle \simeq \langle X_k \rangle \simeq \mathbb{P}^3$, and we get the same contradiction: $\dim[\mathrm{Sec}(X)] \geq 5$. If X_j and X_k are quadric cones with the same vertex P we cannot use Lemma 5, however in this case $T_P(X_j) = \langle X_j \rangle \simeq \mathbb{P}^3 \simeq \langle X_k \rangle = T_P(X_k)$ and their intersection is a line so that $\dim(\langle \mathbb{T}_P \rangle) \geq 5$ and we get a contradiction with Lemma 4.

Note that, on the contrary, if X_j is a smooth quadric or a quadric cone and X_k is a plane we cannot repeat the previous arguments to exclude the existence of quadrics in X. □

Now we give the following:

Corollary 3. *Let $X \subset \mathbb{P}^{n+4}$ be a reducible Veronese surface, according to Definition* 1*, for some $n \geq 1$. Then:*

i) *through any singular point $P \in X$ there passes only* 1, 2 *or* 3 *irreducible components of X and the first case occurs only when P is the vertex of a quadric cone;*
ii) *if P is a singular point of X, not the vertex of a quadric cone, the tangent planes at P to the irreducible components of X passing through P* (2 *or* 3) *are all distinct;*
iii) *if P is a singular point of X which it is the vertex of a quadric cone Γ and there are at least two irreducible components of X passing through P:*
   - *if the components are two, one of them is Γ and the other one is a plane not contained in $\langle \Gamma \rangle$*
   - *if the components are three, one of them is Γ and the other ones are two distinct planes not contained in $\langle \Gamma \rangle$.*

Proof. i) Obviously, by Proposition 2, a singular point $P \in X$ belongs to only one irreducible component X^P of X if and only if X^P is a quadric cone and P is its vertex. In the other cases, let $X_1^P, \dots, X_s^P$ be the irreducible components of X containing P, $s \geq 2$. We can assume that their natural ordering is coherent with the existing ordering considered in Theorem 1. Let T_i be the tangent space of X_i^P at P in $\langle X_i^P \rangle$, $i = 1, \dots, s$.

By Lemma 3, $\dim(\langle T_1 \cup \cdots \cup T_s \rangle) = \dim(\langle \mathbb{T}_P \rangle) \geq \dim(T_1) + s - 1 \geq s + 1$. If $s \geq 4$ we would get a contradiction with Lemma 4, hence $s \leq 3$.

ii) As P is not the vertex of a quadric cone, all the irreducible components of X passing through P are smooth at P by Proposition 2 and they are 2 or 3 by the previous proof. Let T_1, T_2 or T_1, T_2, T_3 be the tangent planes at P to these components, with an ordering coherent with the ordering given by Theorem 1. By Lemma 3, $T_2 \nsubseteq T_1$ and $T_3 \nsubseteq \langle T_1 \cup T_2 \rangle$ so that the planes must be distinct.

iii) By i) we have only one or two other irreducible components of X passing through P and they are planes by Proposition 2. The tangent space at P of Γ is $\langle \Gamma \rangle$, while the tangent spaces at P of the other components coincide with the components themselves, so that they cannot be contained in $\langle \Gamma \rangle$, otherwise we would get a contradiction with Lemma 3 for any possible ordering of these (2 or 3) components coherent with the ordering given by Theorem 1. □

The following lemma is based on property v) of Definition 1 and Corollary 3.

Lemma 6. *Let $X \subset \mathbb{P}^{n+4}$ be a reducible Veronese surface, according to Definition* 1*, for some $n \geq 1$. Let P be a singular point of X such that the union C_P of the irreducible components of X passing through P is a cone, i.e.* (*by Proposition* 2) *the irreducible components of X passing through P are planes and, possibly, a quadric cone with vertex in P. Then if we cut C_P with a generic hyperplane H, not passing through P, the curve $C_P \cap H$ is an Arithmetically Cohen–Macauley* (*in brief ACM*) *scheme.*

Proof. By assumption we know that the local ring of X at P is a Cohen–Macaulay ring; of course it is isomorphic to the local ring of C_P at P. As C_P is a cone over $C_P \cap H$, with vertex P, the local ring of C_P at P is a Cohen–Macaulay ring if and only if $C_P \cap H$ is an ACM scheme. □

Corollary 4. *Let $X \subset \mathbb{P}^{n+4}$ be a reducible Veronese surface, according to Definition* 1*, for some $n \geq 1$. Let P be a singular point of X such that the union C_P of the irreducible components of X passing through P is a cone. Then:*

i) *if P is not the vertex of a quadric cone and there are only two components of X, i.e. two planes, passing through P, then the two planes intersect along a line;*
ii) *if P is not the vertex of a quadric cone and there are three components of X, i.e. three planes, passing through P, then:*
   - *the three planes intersect two by two along three lines passing through P, or*
   - *the three planes intersect along a unique line passing through P and they span a* 3*-dimensional linear space, or*

- *the three planes intersect along a unique line passing through P and they span a 4-dimensional linear space, or*
- *two planes intersect only at P and the third plane cuts the other ones along two lines, passing through P;*

iii) *if P is the vertex of a quadric cone and there is only another component of X, i.e. a plane, passing through P, then the plane cuts the cone only along a line of the cone.*

Proof. Let us apply Lemma 6. In case i) the cone C_P is given by two planes passing through P; if they intersect only at P then the curve $H \cap C_P$ is a pair of disjoint lines in $H \simeq \mathbb{P}^3$ and this is not an ACM scheme.

In case ii) the cone C_P is given by three planes passing through P, and the curve $H \cap C_P$ is a cubic curve reducible into three lines. $H \cap C_P$ is an ACM scheme if and only if it is: a plane cubic given by three lines in generic position or passing through a point ($H \simeq \mathbb{P}^2$) or a space cubic given by a rational comb ($H \simeq \mathbb{P}^3$) or three lines passing through a point and spanning a 3-dimensional linear space ($H \simeq \mathbb{P}^3$). The four possibilities give rise only to the previously described configurations.

In case iii) the cone C_P is given by the union of a quadric cone Γ having vertex at P and a plane passing through P. By Lemma 3 and Corollary 3 iii), the plane is not contained in $\langle \Gamma \rangle$ so that it cuts $\langle \Gamma \rangle$ only at P or along a line L passing through P. If $L \in \Gamma$, then $H \cap C_P$ is a space cubic ($H \simeq \mathbb{P}^3$) given by a smooth conic and a line cutting the conic transversally at some point, a well known ACM scheme. In the other cases $H \cap C_P$ would be the disjoint union of a smooth conic and a line and this is not an ACM scheme. □

4 The main results

In this section we will get a complete classification of reducible Veronese surfaces. First of all we will prove the following theorem.

Theorem 2. *Let $X \subset \mathbb{P}^{n+4}$ be a reducible Veronese surface, according to Definition 1, for some $n \geq 1$, and let us assume that all the irreducible components of X are planes. Then $X = \Sigma_n$.*

Proof. By ii) of Definition 1 we have that X is the union of $n+3$ planes, say $X = \Pi_0 \cup \Pi_1 \cup \cdots \cup \Pi_{n+2}$. By Theorem 1 we can assume that the planes are ordered in such a way that, for any $j \geq 1$, $\Pi_j \cap (\Pi_0 \cup \cdots \cup \Pi_{j-1})$ is a line. Let us call $L_{ij} := \Pi_i \cap \Pi_j$ when the intersection is a line and $Q_{ij} := \Pi_i \cap \Pi_j$ when the intersection is a point. We want to use induction on $n \geq 1$.

Step one. If $n = 1$, $X = \Pi_0 \cup \Pi_1 \cup \Pi_2 \cup \Pi_3$ and we have to prove that $X = \Sigma_1 \subset \mathbb{P}^5$. Let us consider Π_0 and Π_1; by Theorem 1 they intersect along a line L_{01} and $\langle \Pi_0 \cup \Pi_1 \rangle \simeq \mathbb{P}^3$. Let us consider Π_2; by Theorem 1 we know that $\Pi_2 \cap \langle \Pi_0 \cup \Pi_1 \rangle$ is a line L. By Lemma 2 ii) we have that $\Pi_2 \cap \Pi_0 \neq \emptyset$ and $\Pi_2 \cap \Pi_1 \neq \emptyset$, hence $L \cap \Pi_0 \neq \emptyset$ and $L \cap \Pi_1 \neq \emptyset$.

Let us suppose that L intersects Π_0 only at a point $A \notin L_{01}$ and that L intersects Π_1 only at a point $B \notin L_{01}$, so that $\langle \Pi_0 \cup \Pi_1 \cup \Pi_2 \rangle \simeq \mathbb{P}^4$. Then $A = Q_{12}$ and $B = Q_{02}$

are singular points of X. By Corollary 4 i) it is not possible that only two components of X pass through A and B, hence there is another component of X passing through A and there is another component of X passing through B. As X has only four components we have that Π_3 passes through A and B, moreover, by Theorem 1, $\Pi_3 \cap (\Pi_0 \cup \Pi_1 \cup \Pi_2)$ is a line, so that $\Pi_3 \cap (\Pi_0 \cup \Pi_1 \cup \Pi_2) = L$ and $A = Q_{13}$, $B = Q_{03}$. Now let us consider A, for instance, it is a singular point of X and Π_1, Π_2, Π_3 pass through it, but the configuration of these planes contradicts Lemma 4 ii), so that this case is not possible.

Let us suppose that $L = L_{01}$. In this case for any point of L there pass three planes, components of X (this is the maximal number by Corollary 3 i)) intersecting among them only along the line L. By Corollary 4 ii), the three planes belong to the same 3-dimensional linear space, or generate a 4-dimensional linear space. Let us consider the last plane Π_3, it cuts $\Pi_0 \cup \Pi_1 \cup \Pi_2$ along a line L' by Theorem 1, hence L' belongs to Π_0 or to Π_1 or to Π_2 so that in any case $L' \cap L \neq \emptyset$ and for any point in $L' \cap L$ there pass four components of X, but this is a contradiction with Corollary 3 i).

Let us assume that $L \cap L_{01}$ is only one point $P = Q_{02} = Q_{12}$. Through P there pass three planes, components of X (this is the maximal number by Corollary 3 i)), but the configuration of these planes contradicts Lemma 4 ii), so that this case is not possible.

Therefore there is only one possibility: L belongs to one of the two planes Π_0, Π_1 and cuts L_{01} at one point $P = Q_{12}$. We can assume that $L \subset \Pi_0$ by reversing the role of Π_0 and Π_1, if necessary (note that we can change the position of Π_0 and Π_1 in the ordering given by Theorem 1) and we have $L = L_{02}$ and $\langle \Pi_0 \cup \Pi_1 \cup \Pi_2 \rangle \simeq \mathbb{P}^4$. By Theorem 1, $\Pi_3 \cap \langle \Pi_0 \cup \Pi_1 \cup \Pi_2 \rangle$ is a line L' and, by Lemma 2, L' cuts every plane Π_0, Π_1, Π_2, hence it cuts L at some point $A = Q_{03} = Q_{23}$ and it cuts Π_1 at some point $B = Q_{13}$. If $B \notin L_{01}$ then through B would pass only two planes, components of X intersecting only at B and this is a contradiction with Corollary 4 ii). Then $B \in L_{01}$ and $L' = L_{03}$. Note that $B \neq P$ otherwise there would be four components of X passing through P, hence the three lines: L_{01}, $L = L_{02}$, and $L' = L_{03}$ are three lines of Π_0 in general position. Summing up: Π_1, Π_2, Π_3 cut Π_0 along the lines L_{01}, L_{02}, L_{03}, and they cut each other only at the three points $P = Q_{12} = L_{01} \cap L_{02}$, $B = Q_{13} = L_{01} \cap L_{03}$, $A = Q_{23} = L_{02} \cap L_{03}$, so that $X = \Sigma_1$ when $n = 1$.

Step two. Let us assume that $n \geq 2$ and let us define $Y := X \backslash \Pi_{n+2}$. We want to prove that Y is a reducible Veronese surface in $\mathbb{P}^{n'+4}$, according to Definition 1, for $n' := n - 1 \geq 1$. Let us check properties i), …,v).

i) By Theorem 1 we know that $\Pi_{n+2} \cap \langle \Pi_0 \cup \cdots \cup \Pi_{n+1} \rangle$ is a line, hence $\Pi_{n+2} \cap \langle Y \rangle$ is a line. As $n+4 = \dim(\langle X \rangle) = \dim(\langle Y \cup \Pi_{n+2} \rangle) = \dim(\langle Y \rangle) + 2 - \dim(\langle Y \rangle \cap \Pi_{n+2}) = \dim(\langle Y \rangle) + 1$ (we are assuming that $\dim(\emptyset) = -1$), we get that $\dim(\langle Y \rangle) = n + 3 = n' + 4$, so that Y is a non-degenerate, reduced, reducible surface of pure dimension 2 in $\mathbb{P}^{n'+4}$.

ii) $\deg(Y) = \deg(X) - 1 = n + 2 = n' + 3$, $\mathrm{cod}(Y) = n' + 2$.

iii) $\dim[\mathrm{Sec}(Y)] \leq \dim[\mathrm{Sec}(X)] \leq 4$.

iv) Y is a set of ordered planes $\Pi_0, \ldots, \Pi_{n+1}$ in $\mathbb{P}^{n'+4}$ such that:

- $\langle \Pi_0 \cup \cdots \cup \Pi_{n+1} \rangle = \mathbb{P}^{n'+4}$ by the previous check of i),
- for any $j \geq 1$, $\dim(\Pi_j \cap \langle \Pi_0 \cup \cdots \cup \Pi_{j-1} \rangle = 1$ by Theorem 1 (recall that we have ordered all the components of X according to this theorem).

Hence we can apply Corollary 1 and we get that Y is connected in codimension 1.

v) To prove that Y is locally Cohen–Macaulay we have to check all points of Y, obviously we have to check only the points of $Y \cap \Pi_{n+2}$ because for all other points of Y the property follows from the fact that X is locally Cohen–Macaulay.
Let P be a point of $Y \cap \Pi_{n+2}$ and let us assume that there exists only one component $\Pi_i \subset Y$ such that $P \in \Pi_i \cap \Pi_{n+2}$. As X is locally Cohen–Macaulay at P, by Corollary 4 i), we have that Π_i intersects Π_{n+2} along a line passing through P, so that when we delete Π_{n+2} we have that P is a smooth point of Y.
Let us assume that there are two components $\Pi_i, \Pi_j \subset Y$ such that $P \in \Pi_i \cap \Pi_j \cap \Pi_{n+2}$ (two is the maximal number by Corollary 3 i)). As X is locally Cohen–Macaulay at P, by Corollary 4 ii), we have the following possibilities:

- the three planes intersect two by two along three lines passing through P; in this case when we delete Π_{n+2} we get that Π_i intersect Π_j along a line passing through P and Y is locally Cohen–Macaulay at P (see also the proof of Corollary 4 ii));
- the three planes intersect along a unique line passing through P and they span a 3-dimensional or a 4-dimensional linear space; in these cases we can argue as in the previous case and Y is locally Cohen–Macaulay at P;
- Π_i (or Π_j) and Π_{n+2} intersect only at P and the third plane cuts the other ones along two lines, passing through P; in this case we can argue as in the previous cases and Y is locally Cohen–Macaulay at P;
- Π_i and Π_j intersect only at P and Π_{n+2} cuts the other planes along two lines, passing through P; in this case if we delete Π_{n+2} we have that Y is not locally Cohen–Macaulay at P, so we have to prove that this case is not possible; by contradiction, let us assume that the configuration of Π_i, Π_j and Π_{n+2} is as above; we can assume that $0 \le i < j < n+2$ in the ordering given by Theorem 1, so that $\Pi_j \cap (\Pi_0 \cup \cdots \cup \Pi_i \cup \cdots \cup \Pi_{j-1})$ is a line L passing through P; note that L is contained in at least a plane Π_k among $\Pi_0, \ldots, \Pi_i, \ldots, \Pi_{j-1}$ and that $\Pi_k \neq \Pi_i$ because $\Pi_i \cap \Pi_j = P$ (this implies $j > 1$ because $\Pi_0 \cap \Pi_1$ is a line), then $P \in \Pi_k$, so that we would have four different components of X passing through P and we would have a contradiction with Corollary 3 i).

Step three. Now let us proceed by induction on $n \ge 1$. If $n = 1$ Theorem 2 is true by step one. Let us assume that the theorem is true for any X in $\mathbb{P}^5, \mathbb{P}^6, \ldots, \mathbb{P}^{n+3}$ and let us prove the theorem for $X \subset \mathbb{P}^{n+4}$. As in step two we can decompose $X = Y \cup \Pi_{n+2}$ and we know that Y is a reducible Veronese surface in $\mathbb{P}^{n+3}$ according to Definition 1, by step two. By induction we can say that $Y = \Sigma_{n-1}$ so that $X = \Sigma_{n-1} \cup \Pi_{n+2}$. By Theorem 1 we have that $\Sigma_{n-1} \cap \Pi_{n+2}$ is a line L and, as above, L is contained in at least a plane among $\Pi_0, \ldots, \Pi_{n+1}$.

By contradiction, let us assume that $L \subset \Pi_i$ for some $i > 0$ and let us consider the line L_{0i}. L cannot contain any point $Q_{ij} \in L_{0i}$ ($j = 1, \ldots, n+1$, $j \neq i$) and a fortiori $L \neq L_{0i}$ otherwise we would have four different components of X passing through $Q_{ij} : \Pi_0, \Pi_i, \Pi_j, \Pi_{n+2}$, a contradiction with Corollary 3 i). So that $L \cap L_{0i}$ is a point $P \neq Q_{ij}$ for any $j = 1, \ldots, n+1$, $j \neq i$, and the point $P \in X$ belongs exactly to Π_{n+2}, Π_i, Π_0, but this configuration contradicts Corollary 4 ii) because $\Pi_{n+2} \cap \Pi_i = L$, $\Pi_{n+2} \cap \Pi_0 = P$, $\Pi_i \cap \Pi_0 = L_{0i}$ and $L \cap L_{0i} = P$.

Therefore $L \subset \Pi_0$ (i.e. $L = L_{0(n+2)}$) and to prove that $X = \Sigma_n$ we have only to show that the lines L_{0i} with $i = 1, \ldots, n+1$ and L are in general position on Π_0 i.e. that the curve given by their union has no triple points. But this curve has a triple point if and only if L passes through some point Q_{ij} for some $i, j = 1, \dots, n+1$, $i \neq j$, (recall that $Y = \Sigma_{n-1}$) and we have proved that this is not possible. □

To classify reducible Veronese surfaces containing a quadric we need other lemmas.

Lemma 7. *Let $V = V_1 \cup \cdots \cup V_r \subset \mathbb{P}^N$ be a non-degenerate, reducible, reduced, surface of pure dimension* 2*, whose irreducible components are $V_1, \dots, V_r$. Let $W \subset V$ be a proper subvariety of V such that $W = V_1 \cup \cdots \cup V_\rho$ with $1 \leq \rho < r$. Assume that V and W are connected in codimension* 1*. Then there exists at least a component $V_i \subset V$ with $\rho < i \leq r$ such that $\dim(W \cap V_i) = 1$ and $W \cup V_i$ is connected in codimension* 1*.*

Proof. If $\dim(W \cap V_i) \leq 0$ for any irreducible component $V_i \subset V$ with $\rho < i \leq r$, then $\dim[W \cap (V_{\rho+1} \cup \cdots \cup V_r)] \leq 0$, but this is not possible, otherwise $V \backslash [W \cap (V_{\rho+1} \cup \cdots \cup V_r)]$ would be not connected while we are assuming that V is connected in codimension 1. Hence, by changing the ordering of $V_{\rho+1}, \dots, V_r$ if necessary, we can assume that $\dim(W \cap V_{\rho+1}) \geq 1$. It is not possible that $\dim(W \cap V_{\rho+1}) \geq 2$, otherwise the irreducible surface $V_{\rho+1}$ would be a component of W, so that $\dim(W \cap V_{\rho+1}) = 1$.

Now let us consider $W \cup V_{\rho+1}$. W is connected in codimension 1 by assumptions, $V_{\rho+1}$ is connected in codimension 1 because it is an irreducible surface; as $\dim(W \cap V_{\rho+1}) = 1$ we have that $W \cup V_{\rho+1}$ is connected in codimension 1, too. □

Lemma 8. *Let $X \subset \mathbb{P}^{n+4}$ be a reducible Veronese surface, according to Definition* 1*, for some $n \geq 1$, and let $X_1, \dots, X_r$ be its irreducible components. Let us assume that X contains a quadric Q. Then:*

i) $r = n+2$*;*
ii) *there exists an ordering $X_1, \dots, X_{n+2}$ according to Theorem* 1 *such that $Q = X_1$.*

Proof. i) Recall that, by Proposition 2, Q is the only component of X having degree ≥ 2, so that $n + 3 = \deg(X) = 2 + r - 1$, hence $r = n + 2$.

ii) Let us put $X_1 = Q$. By Lemma 7 there is (at least) another component $X_{\bar{i}} \subset X$ such that $\dim(Q \cap X_{\bar{i}}) = 1$ and $Q \cup X_{\bar{i}}$ is connected in codimension 1, moreover $X_{\bar{i}}$ is a plane. By Corollary 2 $Q \cap X_{\bar{i}}$ is a line. If we put $X_2 = X_{\bar{i}}$ we have that $X_1 \cap X_2 = \langle X_1 \rangle \cap \langle X_2 \rangle$ and the intersection is a line. As $n \geq 1$ we have $r \geq 3$, so that there exists at least another component. Now let us apply Lemma 7 to $X_1 \cup X_2$, which is connected in codimension 1, and there is (at least) another component $X_{\bar{i}} \subset X$ such that $\dim[(X_1 \cup X_2) \cap X_{\bar{i}}] = 1$ and $X_1 \cup X_2 \cup X_{\bar{i}}$ is connected in codimension 1, moreover $X_{\bar{i}}$ is a plane, and so on. By applying Lemma 7 a suitable number of times we get an ordering $X_1, \dots, X_{n+2}$ such that $X_1 = Q$ and, for any $j \geq 2$, $\dim[X_j \cap (X_1, \dots, X_{j-1})] = 1$ and $X_1 \cup \cdots \cup X_j$ is connected in codimension 1.

Let us consider $\langle X_j \rangle \cap \langle X_1 \cup \cdots \cup X_{j-1} \rangle = X_j \cap \langle X_1 \cup \cdots \cup X_{j-1} \rangle$ for any $j \geq 2$ and we have $\dim(X_j \cap \langle X_1 \cup \cdots \cup X_{j-1} \rangle) \geq \dim[X_j \cap (X_1 \cup \cdots \cup X_{j-1})] = 1$. Let us put $a_j := \dim(X_j \cap \langle X_1 \cup \cdots \cup X_{j-1} \rangle)$ for any $j \geq 3$, so that:

$$
\begin{aligned}
&\dim(\langle X_1 \cup X_2 \rangle) = 4 \\
&\dim(\langle X_1 \cup X_2 \cup X_3 \rangle) = \dim(\langle \langle X_1 \cup X_2 \rangle \cup X_3 \rangle) = \dim(\langle X_1 \cup X_2 \rangle) + 2 - a_3 \\
&\dim(\langle X_1 \cup X_2 \cup X_3 \cup X_4 \rangle) = \dim(\langle \langle X_1 \cup X_2 \cup X_3 \rangle \cup X_4 \rangle) = \\
&\quad = \dim(\langle X_1 \cup X_2 \rangle) + 2 - a_3 + 2 - a_4 \\
&\vdots \\
&\dim(\langle X_1 \cup X_2 \cup \cdots \cup X_{n+2} \rangle) = \dim(\langle \langle X_1 \cup X_2 \cup \cdots \cup X_{n+1} \rangle \cup X_{n+2} \rangle) = \\
&\quad = \dim(\langle X_1 \cup X_2 \rangle) + 2 - a_3 + 2 - a_4 + \cdots + 2 - a_{n+2} = \\
&\quad = 4 + 2n - \sum_{j=3}^{n+2} a_j = n + 4.
\end{aligned}
$$

Hence $\sum_{j=3}^{n+2} a_j = n$. As $a_j \geq 1$ for any $j \geq 3$ we have in fact $a_j = 1$ for any $j \geq 3$, so that $1 = \dim(X_j \cap \langle X_1 \cup \cdots \cup X_{j-1} \rangle) = \dim[X_j \cap (X_1 \cup \cdots \cup X_{j-1})]$ for any $j \geq 3$ (the case $j = 2$ was considered previously) and $X_j \cap \langle X_1 \cup \cdots \cup X_{j-1} \rangle$ is obviously a line.

To prove Lemma 8 ii) now we have to show that $X_j \cap \langle X_1 \cup \cdots \cup X_{j-1} \rangle = X_j \cap (X_1 \cup \cdots \cup X_{j-1})$ for any $j \geq 2$. As above, the case $j = 2$ was considered previously, so we can assume $j \geq 3$ and recall that X_j is a plane. As $X_j \cap \langle X_1 \cup \cdots \cup X_{j-1} \rangle \supseteq X_j \cap (X_1 \cup \cdots \cup X_{j-1})$ and $X_j \cap \langle X_1 \cup \cdots \cup X_{j-1} \rangle$ is a line we have only to show that $X_j \cap (X_1 \cup \cdots \cup X_{j-1})$ is a line. As $\dim[X_j \cap (X_1 \cup \cdots \cup X_{j-1})] = 1$ there exists at least one component X_i, with $1 \leq i \leq j-1$, such that $\dim[X_j \cap X_i)] = 1$, hence $X_j \cap X_i$ is a line L_{ij} by Corollary 2. Moreover there are no other points $P \in X_j \cap (X_1 \cup \cdots \cup X_{j-1})$, $P \notin L_{ij}$, otherwise X_j would be contained in $\langle X_1 \cup \cdots \cup X_{j-1} \rangle$ and this is not possible as $\dim(X_j \cap \langle X_1 \cup \cdots \cup X_{j-1} \rangle) = 1$. It follows that, for any $j \geq 3$, $X_j \cap (X_1 \cup \cdots \cup X_{j-1})$ is a line and we are done. □

Now we can conclude this section with the following theorems.

Theorem 3. *Let $X \subset \mathbb{P}^{n+4}$ be a reducible Veronese surface, according to Definition* 1*, for some $n \geq 1$, and let $X_1, \ldots, X_r$ be its irreducible components. Let us assume that X contains a smooth quadric Q. Then $n = 1$, $r = 3$, $X = Q \cup X_1 \cup X_2$, where X_1 and X_2 are planes, and we have only two possibilities:*

a) *Q, X_1, X_2 intersect transversally along a unique line $L = Q \cap X_1 \cap X_2$;*
b) *X_1 and X_2 cut Q along two lines intersecting at a point $P = X_1 \cap X_2$.*

Proof. By Lemma 8 we know that $r = n + 2 \geq 3$ and there exists an ordering $X_1, \ldots, X_{n+2}$ given by Theorem 1 such that $X_1 = Q$, X_i is a plane for any $i \geq 2$ and X_2 cuts Q along a line L. Now let us consider the plane X_3 cutting $Q \cup X_2$ and $\langle Q \cup X_2 \rangle \simeq \mathbb{P}^4$ along a line L' by Theorem 1. We have some cases to consider.

1) Let us assume that $L' \subset X_2$ and $L' \neq L$ so that $L' \cap L$ is a point $\overline{P} \in Q$, then $\langle X_2 \cup X_3 \rangle \simeq \mathbb{P}^3$, $L = \langle X_2 \cup X_3 \rangle \cap \langle Q \rangle$, $\langle Q \cup X_2 \cup X_3 \rangle \simeq \mathbb{P}^5$, and $\overline{P} = Q \cap X_3$

so that $\langle Q \cup X_3 \rangle = \langle Q \cup X_2 \cup X_3 \rangle \simeq \mathbb{P}^5$. This case is not possible, in fact, let P be a generic point in $\langle Q \cup X_3 \rangle$; note that, in particular, this means that $P \notin \langle Q \rangle \cup X_3$ and $P \notin \langle T_{\overline{P}}(Q) \cup X_3 \rangle \simeq \mathbb{P}^4$. Let us consider the 3-dimensional linear space $\Lambda_P := \langle P \cup X_3 \rangle \subset \langle Q \cup X_3 \rangle \simeq \mathbb{P}^5$. We have that $\Lambda_P \cap \langle Q \rangle$ is a line L_P passing through $\overline{P}$ and that there exists (at least) another point $P' \in Q$ on L_P with $\overline{P} \neq P'$; recall that $P \notin \langle T_{\overline{P}}(Q) \cup X_3 \rangle$ so that the line L_P is not tangent to Q. Now the line PP' $\in \Lambda_P$ cuts X_3 at some point $P'' \neq \overline{P}$ (otherwise $L_P = PP'$ and $P \in \langle Q \rangle$) so that $P \in \mathrm{Sec}(Q \cup X_3) \subset \mathrm{Sec}(X)$. It follows that the generic point of $\langle Q \cup X_2 \cup X_3 \rangle \simeq \mathbb{P}^5$ is contained in $\mathrm{Sec}(X)$, hence $\dim[\mathrm{Sec}(X)] \geq 5$ and we get a contradiction with iii) of Definition 1.

2) Let us assume that $L' \subset X_2$ and $L' = L$. By contradiction let us assume that there exists another plane X_4 in X. Then $X_4 \cap (Q \cup X_2 \cup X_3)$ is a line L'', but L'' cannot be contained in X_2 or in X_3 otherwise we would have four components of X passing through a point and this is not possible by Corollary 3 i), hence $L'' \subset Q$. Analogously we have $L'' \cap L = \emptyset$, but in this case X_4 must intersect X_2 at some point P by Lemma 2 ii), so that $X_4 = \langle P \cup L'' \rangle$ would be contained in $\langle Q \cup X_2 \cup X_3 \rangle$ and this is not possible by Lemma 8 ii). Hence there are only two planes in X and we get a).

3) Let us assume that $L' \subset Q$ and that $L \cap L' = \emptyset$. Then $X_3 \cap X_2$ would be a point P by Lemma 2 ii) and we would get a contradiction by arguing as above: $X_3 = \langle L' \cup P \rangle$ would be contained in $\langle Q \cup X_2 \rangle$.

4) Let us assume that $L' \subset Q$ and that $L \cap L'$ is a point P and, by contradiction, let us assume that there exists another plane X_4 in X. Then $X_4 \cap (Q \cup X_2 \cup X_3)$ is a line L''. If $L'' \subset Q$, $L'' \neq L$, $L'' \neq L'$ then $X_4 \cap X_2 = \emptyset$ or $X_4 \cap X_3 = \emptyset$ and this is not possible by Lemma 2 ii), on the other hand if $L'' = L$ or $L'' = L'$ we would have four components of X passing through a point and this is not possible by Corollary 3 i). So that $L'' \nsubseteq Q$ and $L'' \subset X_2$ or $L'' \subset X_3$. Now let us suppose that $L'' \subset X_2$ (the other case is similar), if $P \notin L''$ then $X_4 \cap X_3 = \emptyset$ and this is not possible by Lemma 2 ii), on the other hand if $P \in L''$ we would have four components of X passing through a point and this is not possible by Corollary 3 i). Hence there are only two planes in X and we get b).

To complete the proof of Theorem 3 now we have to prove that the surfaces X in cases a) and b) are reducible Veronese surfaces according to Definition 1: i), ii) and iv) are obvious; for iii) let us remark that $\mathrm{Sec}(X)$ is the union of a finite number of linear spaces of dimension ≤ 4; for v) we can apply Lemma 1. □

Theorem 4. *Let $X \subset \mathbb{P}^{n+4}$ be a reducible Veronese surface, according to Definition 1, for some $n \geq 1$, and let $X_1, \ldots, X_r$ be its irreducible components. Then none of the components of X can be a quadric cone.*

Proof. By contradiction, let us suppose that X contains a quadric cone Γ of vertex P_Γ. By Lemma 8 we know that $r = n + 2 \geq 3$ and there exists an ordering $X_1, \ldots, X_{n+2}$ such that $\Gamma = X_1$, the other components are planes and $X_2 \cap \Gamma$ is a line L passing through P_Γ. Let us consider the plane X_3 and let us remark that $P_\Gamma \notin X_3$, in fact the union of the tangent spaces to Γ and X_2 at P_Γ spans the 4-dimensional linear space $\langle \Gamma \cup X_2 \rangle$ and $X_3 \notin \langle \Gamma \cup X_2 \rangle$ by Theorem 1, so that, if $P_\Gamma \in X_3$, we would get a contradiction with

Lemma 4 for $P = P_\Gamma$.

On the other hand we know that $X_3 \cap (\Gamma \cup X_2)$ is a line L' by Theorem 1. As $P_\Gamma \notin X_3$ we have that $L' \nsubseteq \Gamma$, so that $L' \subset X_2$ and it cuts Γ only at a point $\overline{P} \in L$, $\overline{P} \neq P_\Gamma$. Hence X_3 and Γ are in the same configuration as X_3 and Q in Case 1) of Theorem 3, so that we can argue as above and we can prove that this case is not possible. Therefore X_3 does not exist and we get a contradiction as $r \geq 3$. □

Remark 3. The above Theorems 2, 3 and 4, taking into account Proposition 2, give a complete classification of the reducible Veronese surfaces according to Definition 1. It follows that the generic surfaces S_n, embedded in $\mathbb{P}^{n+4}$, introduced by Floystad in [5], are in fact surfaces Σ_n for any $n \geq 2$. If $n = 2$ the proof was made in Section 2. If $n \geq 3$ we have only to check that any generic S_n satisfies Definition 1: in [5] it is proved that S_n is non-degenerate and that iii) and v) hold; from v) it follows that S_n is reduced, of pure dimension 2, and that iv) holds (see Remark 1); ii) follows from direct calculation as in Section 2; to have i) it suffices to show that S_n is reducible, if not, from the classification of irreducible, reduced surfaces of minimal degree (see the beginning of the proof of Proposition 2) it would follow that $\deg(S_n) \leq 4$, while $\deg(S_n) \geq 6$ as $n \geq 3$.

Remark 4. Reducible Veronese surface X are not locally complete intersections. In fact let us consider any triple point $P \in X$ and let Y_p be any generic hyperplane section of X passing through P. If X is a locally complete intersection at P then Y_p is a locally complete intersection at P too (see for instance [2, Theorem 2.3.4]). If $X = \Sigma_n$ then Y_p is the union of 3 lines passing through P, spanning a 3-dimensional linear space. If X is one of the cases a), b) of Theorem 3 then Y_P is the union of a smooth conic and two lines passing through P, spanning a 4-dimensional linear space. In any case Y_P is not a locally complete intersection at P.

Remark 5. Reducible Veronese surfaces are not even locally Gorenstein. Let X, P, Y_P be as in Remark 4. If X is locally Gorenstein at P then the dualizing sheaf ω_X is free at P and it has rank 1 (see [3, p. 532]). By adjunction we have that $\omega_{Y_P} = (\omega_X + H)_{|Y_P}$ where H is the Cartier divisor of X corresponding to Y_P (see Lemma 1.7.6 of [1]), so that ω_{Y_P} is free at P and it has rank 1 too. But this is not possible: let $f : \overline{Y_P} \longrightarrow Y_P$ be the normalization of Y_P; note that f is a triple unramified covering locally at P. The conductor sheaf $\mathcal{C}$ of $\mathcal{O}_{\overline{Y_P},P}$ in $\mathcal{O}_{Y_P,P}$ is the maximal ideal of $\mathcal{O}_{Y_P,P}$, hence $\dim_{\mathbb{C}}(\mathcal{O}_{Y_P,P}/\mathcal{C}) = 1$, on the other hand $\dim_{\mathbb{C}}(\mathcal{O}_{\overline{Y_P},P}/\mathcal{O}_{Y_P,P}) = 2$ and this is a contradiction because $\dim_{\mathbb{C}}(\mathcal{O}_{\overline{Y_P},P}/\mathcal{O}_{Y_P,P}) = \dim_{\mathbb{C}}(\mathcal{O}_{\overline{Y_P},P}/\mathcal{C}) + \dim_{\mathbb{C}}(\mathcal{O}_{Y_P,P}/\mathcal{C}) = 2 + 1 = 3$.

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